# Towards Phonon-Like Excitations of Instanton Liquid

S.V. Molodtsov[1], A.M. Snigirev[2], G.M. Zinovjev[3]

[1] *State Research Center, Institute of Theoretical and Experimental Physics, Moscow, Russia*
[2] *Nuclear Physics Institute, Moscow State University, Moscow, Russia*
[3] *Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine.*



**Abstract**

The phonon-like excitations of (anti)-instanton ($\bar{I}I$) liquid due to adiabatic variations of vacuum wave functions are studied in this paper. The kinetic energy term is found and the proper effective Lagrangian for such excitations is evaluated. The properties of their spectrum, while corresponding masses are defined by $\Lambda_{QCD}$, are investigated.

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The model of (anti)-instanton liquid correctly seizes many nonperturbative phenomena and important vacuum features such as chiral symmetry breaking, the presence of gluon condensate and topological susceptibility [1],[2]. It is usually supposed that the corresponding functional integral in this approach is saturated by quasi-classical configurations close to the exact solutions of the Yang-Mills equations (the Euclidean solutions called the (anti)-instantons) and the wave function of vacuum, being homogeneous in metric space, is properly reproduced by averaging over their collective coordinates. In the $\bar{I}I$ liquid approach one takes the superposition ansatz of the pseudo-particle (PP) fields as one of the simplest relevant approximations to the 'genuine' vacuum configuration

$$A_\mu(x) = \sum_{i=1}^{N} A_\mu(x;\gamma_i)\ . \tag{1}$$

Here $A_\mu(x;\gamma_i)$ denotes the field of a singled (anti)-instanton in singular gauge with $4N_c$ (for the $SU(N_c)$ group) coordinates $\gamma = (\rho,\ z,\ \Omega)$, of size $\rho$ with the coordinate of its centre $z$, $\Omega$ as its colour orientation and

$$A^a_\mu(x;\gamma) = \frac{2}{g}\ \Omega^{ab}\bar{\eta}_{b\mu\nu}\frac{y_\nu}{y^2}\frac{\rho^2}{y^2+\rho^2}\ ,\quad y = x - z\ , \tag{2}$$

where $\eta$ is the 't Hooft symbol [3]. For anti-instanton $\bar{\eta} \to \eta$ (making the choice of the singular gauge allows us to sum up the solution preserving the asymptotic behaviour). For simplicity we shall not introduce different symbols for instanton and anti-instanton, and then in the superposition of Eq.(1) $N$ implies the PP total number in the 4-volume $V$ system with the density $n = N/V$. The action of the instanton liquid model is introduced by the following functional

$$\langle S\rangle = \int d^4x \int d\rho\ n(\rho)s(\rho)\ . \tag{3}$$

The integration should be performed over the system volume along with averaging the one instanton action $s(\rho)$ weighted by instanton size distribution function $n(\rho)$. Then an action per

[1]molodtsov@vitep5.itep.ru
[2]snigirev@lav.npi.msu.su
[3]gezin@physik.uni-bielefeld.de, gezin@ap3.bitp.kiev.ua

one instanton is given by well-known expression

$$s_1(\rho) = \beta(\rho) + 5\ln(\Lambda\rho) - \ln\widetilde{\beta}^{2N_c} + \beta\xi^2\rho^2 \int d\rho_1 n(\rho_1)\rho_1^2 \ , \qquad (4)$$

with the Gell-Mann-Low beta function $\beta(\rho) = -\ln C_{N_c} - b\ln(\Lambda\rho)$ , $\Lambda = \Lambda_{\overline{MS}} = 0.92\Lambda_{P.V.}$ , and constant $C_{N_c}$ depending on the regularization scheme, here $C_{N_c} \approx \frac{4.66\ \exp(-1.68N_c)}{\pi^2(N_c-1)!(N_c-2)!}$, $b = \frac{11}{3}N_c$ , and the parameters $\beta = \beta(\bar{\rho})$ and $\widetilde{\beta} = \beta + \ln C_{N_c}$ are the $\beta$ function values at the fixed quantity of $\bar{\rho}$ (average instanton size).

Some terms (inperfect $\rho$ dependence) of Eq.(4) could be obtained in the classical Yang-Mills theory with one-loop (quantum) corrections taken into account and resulting in a modification of coupling constant $g$ at the distinct scales. Indeed, the first term is the one instanton action $8\pi^2/g^2$ with the $\rho$-dependence of $g$ corrected. The last term of Eq.(4) describes the pair interaction of PP's in the instanton ensemble with the constant $\xi$ characterizing, in a sense, the intensity of interaction $\xi^2 = \frac{27}{4}\frac{N_c}{N_c^2-1}\pi^2$. The smallness of characteristic instanton liquid parameter $n\rho^4$ ('packing fraction') allows us to drop out the $\rho$ dependence of the $\beta$-function. The logarithmic terms correspond to the pre-exponential factor contribution to the functional integral and are of a genuine quantum nature. In the second term the $\rho^5$ factor makes the integration elements $d\rho$ and $d^4z$ dimensionless. Finally, the third term is a square root of the one instanton action $\widetilde{\beta}$ raised to the power $4N_c$. The latter is just the zero mode number of the one instanton solution and the corresponding $\rho$ dependence may be again omitted because of small logarifmic contribution.

Taking the exponential form for the distribution function over the instanton action $n(\rho) \sim e^{-s_1(\rho)}$, we obtain directly from Eq.(3) the self-consistent description of equilibrium state of instanton liquid with the well-known ground state [4]

$$\mu(\rho) = \rho^{-5}\widetilde{\beta}^{2N_c} e^{-\beta(\rho)-\nu\rho^2/\overline{\rho^2}} \ , \nu = \frac{1}{2}(b-4) \ , \left(\overline{\rho^2}\right)^2 = \frac{\nu}{\beta\xi^2 n} \ , \ n = \int d\rho\, n(\rho) \ , \ \overline{\rho^2} = \int d\rho\, \rho^2 n(\rho)/n.$$

The distribution $\mu(\rho)$ has obvious physical meaning, namely, the quantity $d^4x\ d\rho\ \mu(\rho)$ is proportional to the probability to find an instanton of size $\rho$ at some point of a volume element $d^4x$. At small $\rho$ the behaviour of distribution function is stipulated by the quantum-mechanical uncertainty principle preventing a solution being compressed at a point (radiative correction). At large $\rho$ the constraint comes from the repulsive interaction between the PP's which is amplified with (anti)-instanton size increasing.

Deriving Eq.(3) we should average over the instanton positions in a metric space. It is clear that the characteristic size, which has to be taken into account, should be larger enough than the mean instanton size $\bar{\rho}$. But it should not be too large because the far ranged fragments of instanton liquid are not causally dependent. The vacuum wave function is expected to be homogeneous on this scale $L \geq \bar{R}$ ($\bar{R}$ is an average distance between the PP's). Let us remind that each PP contributes to the functional integral with the weight factor proportional to $\sim 1/V$, $V = L^4$. The characteristic configuration saturating the functional integral is taken as the superposition (1) with $N$ pseudoparticles in the volume $V$. If one supposes that the PP

[4] This argument corresponds to the maximum principle of [2]. Approaching the functional (3) as a local form $\langle S\rangle = \int d\rho\ s_1(\rho)n(\rho)/n$ where $s_1(\rho) = \beta(\rho) + 5\ln(\Lambda\rho) - \ln\widetilde{\beta}^{2N_c} + \beta\xi^2\rho^2 n\overline{\rho^2}$, using $n(\rho) = Ce^{-s(\rho)}$ with the constant $C$ as a distribution function (actually it makes the problem self-consistent because an equillibrium distribution function should be dependent on an action only) and taking the variation of $\langle S\rangle - \langle S_1\rangle = \int d\rho\ \{s(\rho) - s_1(\rho)\}e^{-s(\rho)}/n$ over $s(\rho)$ one may come to the result $s(\rho) = s_1(\rho) + const$ keeping into the mind an arbitrary normalization.

number in an ensemble is still appropriate to consider them separately, then denoting $\triangle N(\rho_i)$ as the PP number of size $\rho \in (\rho_i,\ \rho_i + \triangle\rho)$, $K$ as the number of partitions within the interval $(\rho_i,\ \rho_f)$, Eq.(1) may be rewritten in the following form

$$A_\mu(x) = \sum_{i=1}^{K} \sum_{j=1}^{\triangle N(\rho_i)} A_\mu(x; i, \gamma_j) \ , \tag{5}$$

where $A_\mu(x; i, \gamma_j)$ is the (anti)-instanton solution with the calibrated size and $\gamma = (z,\ \Omega)$ stands for the coordinate of its centre and colour orientation. By definition $\sum_{i=1}^{K} \triangle N(\rho_i) = N$. Further, introducing the distribution function $n(\rho) = \frac{\triangle N(\rho)}{\triangle\rho}\frac{1}{V}$, and normalizing it as $\sum_{i=1}^{K} n(\rho_i)\ \triangle\rho\ V = N$ (in the continual limit $\triangle\rho \to 0$ it is valid $V \int d\rho\ n(\rho) = N$) one can calculate the classical action $S_c = \frac{1}{4}\int d^4x\ G_{\mu\nu}^2$ of this configuration averaging over the instanton positions in the metric and colour spaces. As a result (with the superposition ansatz Eq.(1)) one instanton actions and the PP pair interactions only contribute to the average system action

$$\langle S_c \rangle = \prod_{i=1}^{N} \int \frac{d^4 z_i}{V} d\Omega_i\ S_c\ = \int d^4 z \int d\rho\ n(\rho) \left\{ \frac{8\pi^2}{g^2} + \frac{8\pi^2}{g^2}\beta\xi^2\rho^2 \int d\rho_1 n(\rho_1)\rho_1^2 \right\} \ , \tag{6}$$

where $d\Omega$ is a measure in the colour space with the unit normalization. As above mentioned $\langle S_c \rangle$ including one loop corrections will then contribute to the functional integral.

It is easy to understand that Eq.(3) describes properly even non-equilibrium states of the instanton liquid when the distribution function $n(\rho)$ does not coincide with the ground state one $\mu(\rho)$. Moreover, it allows us to generalize Eq.(3) for the non-homogeneous liquid, when the size of the non-homogeneity obeys the obvious constraint $\lambda \geq L \geq \bar{\rho}$.

In what follows, we study the excitations of $\bar{I}I$ liquid induced by adiabatic dilatational deformations of the instanton solutions. Then, as the configurations saturating the functional integral we consider not the instanton solution itself but the quasizero modes which are parametrically very close in the functional space (a direction does exist where the action varies slowly) to the zero modes. The guiding idea of selecting a deformation originates from transparent observation. The deformations measured in units of the action $\frac{dq\ dp}{2\pi\hbar}$ (here $q,\ p$ are the generalized coordinate and momentum) have a physical meaning only. However, the instantons are characterized by 'static' coordinates $\gamma$ and, therefore, need to appoint the conjugated momenta. It looks quite natural for the variable, for example, $\rho$ to introduce those as $\dot{\rho} = d\rho/dx_4$.

Let us calculate first of all the corrections for the one-instanton action. Dealing with superposition ansatz Eq.(1) again, one should include additional contribution to the chromoelectric field

$$G'^a_{\mu\nu} = G^a_{\mu\nu} + g^a_{\mu\nu} \ . \tag{7}$$

with the first term of strength tensor (s.t.) corresponding to the contribution generated by the instanton profile

$$G^a_{\mu\nu} = -\frac{8}{g}\frac{\rho^2}{(y^2+\rho^2)^2}\left(\frac{1}{2}\bar{\eta}_{a\mu\nu} + \bar{\eta}_{a\nu\rho}\frac{y_\mu y_\rho}{y^2} - \bar{\eta}_{a\mu\rho}\frac{y_\nu y_\rho}{y^2}\right) \ ,$$

and in adiabatic approximation the corrections have the form

$$g^a_{4i} \approx \frac{\partial A^a_i}{\partial\rho}\dot{\rho} = \frac{4}{g}\bar{\eta}_{ai\nu}\ \frac{y_\nu\ \rho}{(y^2+\rho^2)^2}\ \dot{\rho}, \quad g^a_{ij} = 0 \ , \quad g^a_{i4} = -g^a_{4i} \ , \quad i,j = 1,2,3 \ .$$

Here we have justifiably ignored the terms of order $O(\ddot{\rho})$, $O(\dot{\rho}^2)$. The adiabatic constraint $g^a_{\mu\nu} \ll G^a_{\mu\nu}$ means that the variation of instanton size is much smaller than the characteristic transformation scale of the PP field, $\dot{\rho} \ll O(1)$. Then calculating the corrections for the action, it is reasonable to take out $\dot{\rho}$ beyond the integral and the one instanton contribution to the action turns out to be

$$s_c = \frac{1}{4} \int d^4x \; G'^2_{\mu\nu} \simeq \frac{8\pi^2}{g^2} + C \; \dot{\rho} + \frac{\kappa_{s.t.}}{2} \; \dot{\rho}^2, \tag{8}$$

where $\dot{\rho}$ should be taken as the mean rate of slow solution deformation at a characteristic instanton lifetime $\sim \rho$. For simplicity, one may take it in the centre of the instanton $\dot{\rho}(z)$. The constant $C = 0$ (because the first variation of the action $\delta S/\delta A$ for the solution itself equals to zero). For the 'kinematical' $\kappa$-term we have

$$\kappa_{s.t.} = \frac{12\pi^2}{g^2} \; . \tag{9}$$

The overt $\rho$ dependence of $\kappa$ is lacking because of the scale invariance. It arises with the renormalization of the coupling constant (in a regular gauge the result is the same). Being within the ansatz (1) we have considered only the corrections induced by the variation of strength tensors, but not those resulting from a possible variation of fields (2). Bearing in mind the form of potentials in regular ($r.g.$) and singular ($s.g.$) gauges

$$A^a_\mu = \frac{1}{g} \eta_{a\mu\nu} \; \partial_\nu \ln(y^2 + \rho^2) \;\; (\text{ r.g.}) \; ; \;\; A^a_\mu = -\frac{1}{g} \bar{\eta}_{a\mu\nu} \; \partial_\nu \ln\left(1 + \frac{\rho^2}{y^2}\right) \;\; (\text{ s.g.}) \; , \tag{10}$$

we find the adiabatic corrections $A'_\mu = A_\mu + a_\mu$ as follows:

$$a^a_\mu = \frac{2}{g} \eta_{a\mu4} \; \frac{\rho}{y^2 + \rho^2} \; \dot{\rho} \qquad (\text{r.g.}) \; . \tag{11}$$

The substitution $\eta \to -\bar{\eta}$ brings about the transition from regular gauge to a singular one. Using the admixture $g^a_{\mu\nu}$ of chromoelectric and chromomagnetic fields generated by the $a^a_\mu$ to saturate the functional integral as in Eq.(7) we drop the terms of higher order than $O(\dot{\rho})$ out. Thus, we come to the result for the 'kinematical' term $\kappa$ as

$$\kappa = \frac{32\pi^2}{g^2} \; . \tag{12}$$

But within the accuracy taken at calculating Eq.(3) we are permitted to fix $\kappa$ at some point as $\kappa(\bar{\rho})$.

Analysing the variations of the functional integration result when having calculated for the quasizero mode in the adiabatic approximation we see that because of the kinetic energy smallness it is certainly permitted to neglect its impact on the pre-exponential factor, which is small itself. The inverse infuence of the pre-exponential factor on the 'kinematical' term is negligible as well. Thus, going the way which we have already passed through while calculating Eq.(3) we receive whithin the required order of accuracy [5]

$$\langle S \rangle = \int d^4z \int d\rho \; n(\rho) \; \left\{\frac{1}{2} \kappa \; \dot{\rho}^2 + s(\rho)\right\} \; . \tag{13}$$

[5] In principle, such an integral can be calculated exactly [4]. For the case at hand the types of deformations (quasizero modes) may be simply counted in terms of the instanton solution. They are induced by the variations of the instanton size, the changes of its position and colour orientation. In fact, there is one more type of the quasizero modes related to two far distant instantons but it has already been studied in [4].

It is important to remark here that the contribution of PP pair interacting term averaged over the colour orientation will be [2]

$$\langle S(12)\rangle = \int \frac{d^4 z_1}{V} \int \frac{d^4 z_2}{V}\, \overline{U}_{int}(12)\ , \tag{14}$$

$$\overline{U}_{int}(12) = \frac{8\pi^2}{g^2}\frac{N_c}{N_c^2-1}\int d^4x\ \frac{[7y_1^2y_2^2-(y_1y_2)^2]\ \rho_1^4\rho_2^4}{y_1^4(y_1^2+\rho_1^2)^2\ y_2^4(y_2^2+\rho_2^2)^2}\ ,\quad y_i = x - z_i\ ,\quad \rho_i=\rho_i(z_i)\ ,\quad i=1,2.$$

The way how one instanton of the size $\rho_1$ affects another of the size $\rho_2$ is estimated by the following integral

$$\int \frac{d^4 z_1}{V}\overline{U}_{int}(12) \simeq \frac{8\pi^2}{g^2}\frac{\xi^2}{V}\ \rho_1^2(z_2)\ \rho_2^2(z_2)\ .$$

Indeed, the adiabatic constraint makes it possible to rescale the integration variable $\frac{dz_1}{\rho_1} = d\left(\frac{z_1}{\rho_1}\right) + \frac{z_1}{\rho_1^2} d\rho_1 \approx d\left(\frac{z_1}{\rho_1}\right)$. Besides, we appraise the magnitude of a slowly varying function $\rho_1(z_1)$ at the point where the integrand reaches its maximum. An analysis shows that the points $z_1$ and $z_2$ are not much separated as $|z_1-z_2| \sim \max\{\rho_1,\rho_2\}$, then the $\rho_1(z_2)$ looks like a quite suitable choice (the contact interaction) under the adiabaticity condition. In particular, if the PP sizes do not vary we reproduce the well-known result [2] $\langle S_{int}(12)\rangle = \frac{8\pi^2}{g^2}\ \frac{\xi^2}{V}\ \rho_1^2\rho_2^2$ for such a contribution. The integral $\int d\rho\ \rho^2\ n(\rho)/n$ is approximated by $\overline{\rho^2}$ in the approach we are going herein along (what is intuitively clear and can be easily argued) and eventually it results in the self-interaction of PP's.

In the Minkowski space the factor in the curle brackets of Eq.(13) might be interpreted as a mechanical system with Lagrangian $\mathcal{L} = \frac{1}{2}\ \kappa\ \dot\rho^2 - U_{eff}(\rho)$ and an action per one instanton might be taken as a 'potential energy' $U_{eff}(\rho) = \beta(\rho) + 5\ln(\Lambda\rho) - \ln\widetilde{\beta}^{2N_c} + \nu\ \frac{\rho^2}{\overline{\rho^2}}$ . In the local vicinity of the potential minimum $\rho_c^2 = \frac{b-5}{2\nu}\ \overline{\rho^2}$, $\left(\frac{U_{eff}}{d\rho} = 0\right)$ the system is oscillating and we have for the frequency (using the configuration corresponding to Eq.(12))

$$m^2 = \frac{4\nu}{\kappa\ \overline{\rho^2}} = \frac{\nu}{\beta\ \overline{\rho^2}} \tag{15}$$

while calculating the second derivative $\frac{d^2U_{eff}(\rho)}{d\rho^2}\ |_{\rho_c} = \frac{4\nu}{\overline{\rho^2}}$.

We have only analysed the deformations in the temporary direction. Those in the spatial directions could be estimated by drawing the same arguments. Thus, the expression for the $\kappa$-term keeps the form obtained above with the only change of rates for the appropriate gradients of function $\rho(x)$, i.e. the substitution $\dot\rho(t) \to \frac{\partial\rho(x)}{\partial x}$ should be performed for such a 'crumpled' instanton. Then the frequency of proper fluctuations might be interpreted as the mass term and the excitations occur to have a phonon-like nature

$$\mathcal{L} = \frac{1}{2}\ \kappa\ [\dot\rho^2 - \nabla\rho\nabla\rho] - U_{eff}(\rho)\ , \tag{16}$$

(the cross-terms $\sim \dot\rho\ \rho'$ equal to zero identically) [6]. The parameters $\bar\rho$ and $\beta(\bar\rho)$ determined (by maximizing the partition function of $\bar I I$ liquid with respect to N [2]) in a self-consistent

[6] It is interesting to remark that the centre of instanton solution may not be shifted since the relevant deformations lead to the singular $\kappa$ (unlike the dilatational mode), whereas the colour coordinate variation $\Omega$ gives the trivial result $\kappa = 0$.

way take the following values $\bar{\rho}\Lambda \approx 0.37$, $\beta \approx 17.5$, $n\ \Lambda^{-4} \approx 0.44$, $(N_c = 3)$ therefore, for the mass term we have $m \approx 1.21\Lambda$. Then the wave length $\lambda_4 \sim m^{-1}$ in the $x_4$-direction is $\lambda_4\Lambda \approx 0.83 \geq \bar{V}^{1/4}\Lambda \approx 0.81 > \bar{\rho}\Lambda$ (the size of the phonon localization in the spatial directions can be arbitrary and may noticeably exceed $\lambda_4$, besides the number $N$ of the PP's forming the excitations might be pretty large).

These numerical values obtained should be taken rather qualitatively illustrating the principle possibility to have the particle-like excitations originated by the quasizero modes. It is clear that striving to go beyond the superposition ansatz one should to take into account at least a medium change of the instanton profile and to develop more realistic description of the instanton interactions. Certainly, what presented here essentially exceeds the corresponding results which one could expect in the 'complete theory'.

In conclusion let us emphasize that we have considered the excitations of the instanton liquid generated by the dilatational instanton deformations and the adiabaticity assumption leads, in principle, to a fully consistent picture. The model itself regulates the most suitable regime of such phonon-like deformations resulting in mass gap generation fixed by $\Lambda_{QCD}$. Apparently, including the quark condensate [7] an intriguing guess is to associate some light hadrons with these phonon-like excitations discovered since the preliminary evaluations of their mass spectrum look quite encouraging. Moreover, the concept of the confining potential for the light quarks in the context of our approach seems simply irrelevant because of a stable phonon nature.

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[7] Since we are working within the adiabatic regime the standard perturbation theory is applicable. Then the preliminary results obtained dealing with the $N_c \to \infty$ approach [5] corroborate that the quark condensate contributes insignificantly $\sim 0.2$ to the $\kappa$ (with the parameter values of the $\bar{I}I$ liquid used in the paper) being in agreement with the well-known theoretical expectations although could play crucial role for producing phonon excitations. However, we understand that in order to be conclusive going to a 'complete theory' we have to include the pion cloud contribution together with an inverse impact of the phonons on the pions in the estimates.

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